

# Properties of Free Baxter Algebras \*

Li Guo

Department of Mathematics and Computer Science  
Rutgers University  
Newark, NJ 07102, USA  
(liguo@newark.rutgers.edu)

## Abstract

The study of free Baxter algebras was started by Rota and Cartier thirty years ago. We continue this study by applying two recent constructions of free Baxter algebras. We investigate the basic structure of a free Baxter algebra, and characterize in detail when a free Baxter algebra is a domain or a reduced algebra. We also describe the nilpotent radical of a free Baxter algebra when it is not reduced.

## 1 Introduction

The study of Baxter operators originated in the work of Baxter [2] on fluctuation theory, and the algebraic study of Baxter operators was started by Rota [14]. Let  $C$  be a commutative ring and let  $\lambda$  be a fixed element in  $C$ . A Baxter algebra of weight  $\lambda$  is a commutative  $C$ -algebra  $R$  together with a  $C$ -linear operator  $P$  on  $R$  such that for any  $x, y \in R$ ,

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy).$$

Baxter algebras have important applications in combinatorics [15, 16] and are closely related to several areas in algebra and geometry, such as differential algebras [11], difference algebras [6] and iterate integrals in geometry [4].

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As in any algebraic system, free Baxter algebras play a central role in the study of Baxter algebras. Even though the existence of free Baxter algebras follows from the general theory of universal algebras, in order to get a good understanding of free Baxter algebras, it is desirable to find concrete constructions of a free Baxter algebra. Two constructions were given in [8, 9], called shuffle Baxter algebras and standard Baxter algebras respectively (see Section 2 for details). The construction of shuffle Baxter algebras is motivated by the shuffle product of iterated integrals [13] and an earlier construction of Cartier [3]. The construction of standard Baxter algebras is motivated by a construction of Rota [14].

In this paper, we apply these two constructions of free Baxter algebras to obtain further information about free Baxter algebras. After a brief discussion of basic properties of free Baxter algebras, we will focus on the investigation of zero divisors and nilpotent elements in a free Baxter algebra. This question has been considered by Cartier [3] and Rota [14, 15] for Baxter algebras of weight one without an identity. In their case, the free Baxter algebras have very good properties. In fact the algebras are often isomorphic to either polynomial algebras or power series algebras. The explicit descriptions of free Baxter algebras obtained in [8, 9] enable us to consider this question for a more general class of Baxter algebras. It is interesting to observe that even if a free Baxter algebra is constructed from an integral domain or a reduced algebra, the free Baxter algebra is not necessarily a domain or a reduced algebra. We show that the obstruction depends on several factors, including the characteristic of the base algebra, the weight of the Baxter algebra, whether or not the Baxter algebra has an identity and whether or not the Baxter algebra is complete. We provide necessary and sufficient conditions for a free Baxter algebra to be a domain or to be reduced (Theorem 4.2 and 4.6), and describe the nilpotent radical when a free Baxter algebra is not reduced (Theorem 4.8).

We first give a brief summary of the concept of Baxter algebras and the two constructions of free Baxter algebras in section 2. In section 3 we study basic properties of free Baxter algebras, such as subalgebras, quotient algebras and limits. In section 4 we study in detail when a free Baxter algebra is a domain or a reduced algebra. We also consider free complete Baxter algebras.

## 2 Free Baxter algebras

For later application, we will describe the constructions of free Baxter algebras [8, 9]. We will also prove some preliminary results.

We write  $\mathbb{N}$  for the additive monoid of natural numbers and  $\mathbb{N}_+$  for the

positive integers. Any ring  $C$  is commutative with identity element  $\mathbf{1}_C$ , and any ring homomorphism preserves the identity elements. For any  $C$ -modules  $M$  and  $N$ , the tensor product  $M \otimes N$  is taken over  $C$  unless otherwise indicated. For a  $C$ -module  $M$  and  $n \in \mathbb{N}_+$ , denote the tensor power

$$M^{\otimes n} = \underbrace{M \otimes \dots \otimes M}_{n \text{ factors}}.$$

## 2.1 Baxter algebras

For a given ring  $C$ , let  $\mathbf{Alg}_C$  denote the category of commutative  $C$ -algebras with an identity. For a given  $\lambda \in C$  and  $R \in \mathbf{Alg}_C$ ,

- a **Baxter operator of weight  $\lambda$  on  $R$  over  $C$**  is a  $C$ -module endomorphism  $P$  of  $R$  satisfying

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \quad x, y \in R; \quad (1)$$

- a **Baxter  $C$ -algebra of weight  $\lambda$**  is a pair  $(R, P)$  where  $R$  is a  $C$ -algebra and  $P$  is a Baxter operator of weight  $\lambda$  on  $R$  over  $C$ .
- a  $C$ -algebra homomorphism  $f : R \rightarrow S$  between two Baxter  $C$ -algebras  $(R, P)$  and  $(S, Q)$  of weight  $\lambda$  is called a **homomorphism of Baxter  $C$ -algebras** if  $f(P(x)) = Q(f(x))$  for all  $x \in R$ .

Denote  $\mathbf{Bax}_{C,\lambda}$  for the category of Baxter  $C$ -algebras of weight  $\lambda$ . If the meaning of  $\lambda$  is clear, we will suppress  $\lambda$  from the notation.

A **Baxter ideal** of  $(R, P)$  is an ideal  $I$  of  $R$  such that  $P(I) \subseteq I$ . The concepts of Baxter subalgebras, quotient Baxter algebras can be similarly defined. It follows from the general theory of universal algebras that limits and colimits exist in  $\mathbf{Bax}_C$  [5],[10, p84], [12, p.210]. In particular, inverse limits and direct limits exist in  $\mathbf{Bax}_C$ .

## 2.2 Shuffle Baxter algebras

For  $m, n \in \mathbb{N}_+$ , define the set of  $(m, n)$ -**shuffles** by

$$S(m, n) = \left\{ \sigma \in S_{m+n} \mid \begin{array}{l} \sigma^{-1}(1) < \sigma^{-1}(2) < \dots < \sigma^{-1}(m), \\ \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \dots < \sigma^{-1}(m+n) \end{array} \right\}.$$

Given an  $(m, n)$ -shuffle  $\sigma \in S(m, n)$ , a pair of indices  $(k, k+1)$ ,  $1 \leq k < m+n$  is called an **admissible pair** for  $\sigma$  if  $\sigma(k) \leq m < \sigma(k+1)$ . Denote  $\mathcal{T}^\sigma$  for the set of admissible pairs for  $\sigma$ . For a subset  $T$  of  $\mathcal{T}^\sigma$ , call the pair  $(\sigma, T)$  a

**mixable  $(m, n)$ -shuffle.** Let  $|T|$  be the cardinality of  $T$ .  $(\sigma, T)$  is identified with  $\sigma$  if  $T$  is the empty set. Denote

$$\bar{S}(m, n) = \{(\sigma, T) \mid \sigma \in S(m, n), T \subset \mathcal{T}^\sigma\}$$

for the set of  $(m, n)$ -**mixable shuffles**.

For  $A \in \mathbf{Alg}_C$ ,  $x = x_1 \otimes \dots \otimes x_m \in A^{\otimes m}$ ,  $y = y_1 \otimes \dots \otimes y_n \in A^{\otimes n}$  and  $(\sigma, T) \in \bar{S}(m, n)$ , the element

$$\sigma(x \otimes y) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(m+n)} \in A^{\otimes(m+n)},$$

where

$$u_k = \begin{cases} x_k, & 1 \leq k \leq m, \\ y_{k-m}, & m+1 \leq k \leq m+n, \end{cases}$$

is called a **shuffle** of  $x$  and  $y$ ; the element

$$\sigma(x \otimes y; T) = u_{\sigma(1)} \hat{\otimes} u_{\sigma(2)} \hat{\otimes} \dots \hat{\otimes} u_{\sigma(m+n)} \in A^{\otimes(m+n-|T|)},$$

where for each pair  $(k, k+1)$ ,  $1 \leq k < m+n$ ,

$$u_{\sigma(k)} \hat{\otimes} u_{\sigma(k+1)} = \begin{cases} u_{\sigma(k)} u_{\sigma(k+1)}, & (k, k+1) \in T \\ u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & (k, k+1) \notin T, \end{cases}$$

is called a **mixable shuffle** of  $x$  and  $y$ .

Fix a  $\lambda \in C$ . Let

$$\mathbb{I}\mathbb{I}\mathbb{I}_C(A) = \mathbb{I}\mathbb{I}\mathbb{I}_{C, \lambda}(A) = \bigoplus_{k \in \mathbb{N}} A^{\otimes(k+1)} = A \oplus A^{\otimes 2} \oplus \dots$$

be the Baxter  $C$ -algebra of weight  $\lambda$  [8] in which

- the  $C$ -module structure is the natural one,
- the multiplication is the **mixed shuffle product**, defined by

$$x \diamond y = \sum_{(\sigma, T) \in \bar{S}(m, n)} \lambda^{|T|} x_0 y_0 \otimes \sigma(x^+ \otimes y^+; T) \in \bigoplus_{k \leq m+n+1} A^{\otimes k} \quad (2)$$

for  $x = x_0 \otimes x_1 \otimes \dots \otimes x_m \in A^{\otimes(m+1)}$  and  $y = y_0 \otimes y_1 \otimes \dots \otimes y_n \in A^{\otimes(n+1)}$ , where  $x^+ = x_1 \otimes \dots \otimes x_m$  and  $y^+ = y_1 \otimes \dots \otimes y_n$ ,

- the weight  $\lambda$  Baxter operator  $P_A$  on  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  is obtained by assigning

$$P_A(x_0 \otimes x_1 \otimes \dots \otimes x_n) = \mathbf{1}_A \otimes x_0 \otimes x_1 \otimes \dots \otimes x_n,$$

for all  $x_0 \otimes x_1 \otimes \dots \otimes x_n \in A^{\otimes(n+1)}$ .

$(\mathbb{I}\mathbb{I}_C(A), P_A)$  is called the **shuffle Baxter  $C$ -algebra on  $A$  of weight  $\lambda$** . When there is no danger of confusion, we often suppress  $\diamond$  in the mixed shuffle product. To distinguish the  $C$ -submodule  $A^{\otimes k}$  of  $\mathbb{I}\mathbb{I}_C(A)$  from the tensor power  $C$ -algebra  $A^{\otimes k}$ , we sometimes denote  $\mathbb{I}\mathbb{I}_C^{k-1}(A)$  for  $A^{\otimes k} \subseteq \mathbb{I}\mathbb{I}_C(A)$ .

For a given set  $X$ , we also let  $(\mathbb{I}\mathbb{I}_C(X), P_X)$  denote the shuffle Baxter  $C$ -algebra  $(\mathbb{I}\mathbb{I}_C(C[X]), P_{C[X]})$ , called the **shuffle Baxter  $C$ -algebra on  $X$  (of weight  $\lambda$ )**. Let  $j_A : A \rightarrow \mathbb{I}\mathbb{I}_C(A)$  (resp.  $j_X : X \rightarrow \mathbb{I}\mathbb{I}_C(X)$ ) be the canonical inclusion map.

**Theorem 2.1** [3, 8]  $(\mathbb{I}\mathbb{I}_C(A), P_A)$ , together with the natural embedding  $j_A$ , is a free Baxter  $C$ -algebra on  $A$  of weight  $\lambda$ . In other words, for any Baxter  $C$ -algebra  $(R, P)$  and any  $C$ -algebra homomorphism  $\varphi : A \rightarrow R$ , there exists a unique Baxter  $C$ -algebra homomorphism  $\tilde{\varphi} : (\mathbb{I}\mathbb{I}_C(A), P_A) \rightarrow (R, P)$  such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{j_A} & \mathbb{I}\mathbb{I}_C(A) \\ & \searrow \varphi & \downarrow \tilde{\varphi} \\ & & R \end{array}$$

commutes. Further, any  $f : A \rightarrow B$  in  $\mathbf{Alg}_C$  extends uniquely to

$$\mathbb{I}\mathbb{I}_C(f) : \mathbb{I}\mathbb{I}_C(A) \rightarrow \mathbb{I}\mathbb{I}_C(B)$$

in  $\mathbf{Bax}_C$ . More precisely,  $\mathbb{I}\mathbb{I}_C(f) = \bigoplus_{n \in \mathbb{N}} f^{\otimes(n+1)}$  with  $f^{\otimes(n+1)} : A^{\otimes(n+1)} \rightarrow B^{\otimes(n+1)}$  being the  $(n+1)$ -th tensor power of the  $C$ -module homomorphism  $f$ . Similarly,  $(\mathbb{I}\mathbb{I}_C(X), P_X)$ , together with the natural embedding  $j_X$ , is a free Baxter  $C$ -algebra on  $X$  of weight  $\lambda$ .

Taking  $A = C$ , we get

$$\mathbb{I}\mathbb{I}_C(C) = \bigoplus_{n=0}^{\infty} C^{\otimes(n+1)} = C \mathbf{1}^{\otimes(n+1)}.$$

where  $\mathbf{1}^{\otimes(n+1)} = \underbrace{\mathbf{1}_C \otimes \dots \otimes \mathbf{1}_C}_{(n+1)\text{-factors}}$ . In this case the mixable shuffle product formula (2) gives

**Proposition 2.2** For any  $m, n \in \mathbb{N}$ ,

$$\mathbf{1}^{\otimes(m+1)} \diamond \mathbf{1}^{\otimes(n+1)} = \sum_{k=0}^m \binom{m+n-k}{n} \binom{n}{k} \lambda^k \mathbf{1}^{\otimes(m+n+1-k)}.$$

## 2.3 Complete shuffle Baxter algebras

We now consider the completion of  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ . Recall that we denote  $\mathbb{I}\mathbb{I}\mathbb{I}_C^k(A)$  for the  $C$ -submodule  $A^{\otimes(k+1)}$  of  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ .

Given  $k \in \mathbb{N}_+$ ,  $\text{Fil}^k \mathbb{I}\mathbb{I}\mathbb{I}_C(A) \stackrel{\text{def}}{=} \bigoplus_{n \geq k} \mathbb{I}\mathbb{I}\mathbb{I}_C^n(A)$ , is a Baxter ideal of  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ . Denote  $\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(A) = \varprojlim \mathbb{I}\mathbb{I}\mathbb{I}_C(A) / \text{Fil}^k \mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ , called the **complete shuffle Baxter algebra on  $A$** , with the Baxter operator denoted by  $\hat{P}$ . It naturally contains  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  as a Baxter subalgebra and is a free object in the category of Baxter algebras that are complete with respect to a canonical filtration defined by the Baxter operator [9]. On other hand, consider the infinite product of  $C$ -modules  $\prod_{k \in \mathbb{N}} \mathbb{I}\mathbb{I}\mathbb{I}_C^k(A)$ . It contains  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  as a dense subset with respect to the topology defined by the filtration  $\text{Fil}^k \mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ . All operations of the Baxter  $C$ -algebra  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  are continuous with respect to this topology, hence extend uniquely to operations on  $\prod_{k \in \mathbb{N}} \mathbb{I}\mathbb{I}\mathbb{I}_C^k(A)$ , making  $\prod_{k \in \mathbb{N}} \mathbb{I}\mathbb{I}\mathbb{I}_C^k(A)$  a Baxter algebra of weight  $\lambda$ , with the Baxter operator denoted by  $\bar{P}$ .

**Theorem 2.3** [9]

1. The map

$$\psi_A : \widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(A) \rightarrow \prod_{k \in \mathbb{N}} \mathbb{I}\mathbb{I}\mathbb{I}_C^k(A), ((x_k^{(n)})_k + \text{Fil}^n \mathbb{I}\mathbb{I}\mathbb{I}_C(A))_n \mapsto (x_k^{(k)})_k$$

is an isomorphism of Baxter algebras extending the identity map on  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ .

2. Given a morphism  $f : A \rightarrow B$  in  $\mathbf{Alg}_C$ , we have the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(A) & \xrightarrow{\psi_A} & \prod_{k \in \mathbb{N}} \mathbb{I}\mathbb{I}\mathbb{I}_C^k(A) \\ \downarrow \widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(f) & & \downarrow \prod_k f_k \\ \widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(B) & \xrightarrow{\psi_B} & \prod_{k \in \mathbb{N}} \mathbb{I}\mathbb{I}\mathbb{I}_C^k(B) \end{array}$$

where  $\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(f)$  is induced from  $\mathbb{I}\mathbb{I}\mathbb{I}_C(f)$  in Theorem 2.1 by taking completion, and  $f_k : \mathbb{I}\mathbb{I}\mathbb{I}_C^k(A) \rightarrow \mathbb{I}\mathbb{I}\mathbb{I}_C^k(B)$  is the tensor power morphism of  $C$ -modules  $f^{\otimes(k+1)} : A^{\otimes(k+1)} \rightarrow B^{\otimes(k+1)}$  induced from  $f$ .

## 2.4 The internal construction

We now describe the construction of a standard Baxter algebra [9], generalizing Rota [14].

For each  $n \in \mathbb{N}_+$ , denote  $A^{\otimes n}$  for the tensor power algebra. Denote the direct limit algebra

$$\overline{A} = \varinjlim A^{\otimes n} = \cup_n A^{\otimes n}$$

where the transition map is given by

$$A^{\otimes n} \rightarrow A^{\otimes(n+1)}, \quad x \mapsto x \otimes \mathbf{1}_A.$$

Note that the multiplication on  $A^{\otimes n}$  here is different from the multiplication on  $A^{\otimes n}$  when it is regarded as the  $C$ -submodule  $\mathbb{I}_C^{n-1}(A)$  of  $\mathbb{I}_C(A)$ . Let  $\mathfrak{A}(A)$  be the set of sequences with entries in  $\overline{A}$ . Thus we have

$$\mathfrak{A}(A) = \prod_{n=1}^{\infty} \overline{A} = \{(a_n)_n \mid a_n \in \overline{A}\}.$$

Define addition, multiplication and scalar multiplication on  $\mathfrak{A}(A)$  componentwise, making  $\mathfrak{A}(A)$  into a  $\overline{A}$ -algebra, with the all 1 sequence  $(1, 1, \dots)$  as the identity. Define

$$P'_A = P'_{A,\lambda} : \mathfrak{A}(A) \rightarrow \mathfrak{A}(A)$$

by

$$P'_A(a_1, a_2, a_3, \dots) = \lambda(0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots).$$

Then  $(\mathfrak{A}(A), P'_A)$  is in  $\mathbf{Bax}_C$ . For each  $a \in A$ , define  $t^{(a)} = (t_k^{(a)})_k$  in  $\mathfrak{A}(A)$  by taking

$$t_k^{(a)} = \otimes_{i=1}^k a_i, \quad a_i = \begin{cases} a, & i = k, \\ 1, & i \neq k. \end{cases}$$

Let  $\mathfrak{S}(A)$  be the Baxter subalgebra of  $\mathfrak{A}(A)$  generated by the sequences  $t^{(a)}$ ,  $a \in A$ .

**Theorem 2.4** [9, 14] *Assume that the annihilator of  $\lambda \in C$  in the  $C$ -module  $\overline{A}$  is zero. The morphism in  $\mathbf{Bax}_C$*

$$\Phi : \mathbb{I}_C(A) \rightarrow \mathfrak{S}(A)$$

*induced by sending  $a \in A$  to  $t^{(a)}$  is an isomorphism. Therefore,  $(\mathfrak{S}(A), P'_A)$  is a free Baxter algebra on  $A$  in the category  $\mathbf{Bax}_C$ . ■*

**Corollary 2.5** *Assume that  $\lambda$  is not a zero divisor in  $C$ . Let  $X$  be a set. The morphism in  $\mathbf{Bax}_C$*

$$\Phi : \mathbb{I}_C(X) \rightarrow \mathfrak{S}(X) \stackrel{\text{def}}{=} \mathfrak{S}(C[X])$$

*induced by sending  $x \in X$  to  $t^{(x)} = (t_1^{(x)}, \dots, t_n^{(x)}, \dots)$  is an isomorphism.*

There is also an internal construction of free complete Baxter algebras.

**Theorem 2.6** [9] Assume that the annihilator of  $\lambda \in C$  in  $\overline{A}$  is trivial. The isomorphism  $\Phi : \mathbb{I}\mathbb{I}_C(A) \rightarrow \mathfrak{S}(A)$  extends to an injective homomorphism of Baxter algebras

$$\widehat{\Phi} : \widehat{\mathbb{I}\mathbb{I}}_C(A) \rightarrow \mathfrak{A}(A).$$

For  $A = C$ , we have  $\mathbb{I}\mathbb{I}_C(C) = \bigoplus_{n \in \mathbb{N}} C \mathbf{1}^{\otimes n}$  and  $\widehat{\mathbb{I}\mathbb{I}}_C(C) = \prod_{n \in \mathbb{N}} C \mathbf{1}^{\otimes n}$  in which the multiplication is given by the equation in Proposition 2.2. Also,  $\overline{C} = \varinjlim C^{\otimes n} \cong C$  and  $\mathfrak{A}(C) = \prod_{n=1}^{\infty} C$  with componentwise addition and multiplication.

**Proposition 2.7** Let  $C$  be a domain and let  $\lambda \in C$  be non-zero. Then for any  $b = \sum_{n=0}^{\infty} b_n \mathbf{1}^{\otimes n} \in \mathbb{I}\mathbb{I}_C(C)$ , we have

$$\Phi(b) = \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \lambda^i b_i \right)_{n \in \mathbb{N}_+} \in \mathfrak{S}(C).$$

The same formula holds for  $\widehat{\Phi}$ .

**Proof:** Since  $\Phi$  is  $C$ -linear, we only to show that, for each  $n \in \mathbb{N}$ ,

$$\Phi(\mathbf{1}^{\otimes n}) = \left( \binom{k-1}{n} \lambda^n \right)_k. \quad (3)$$

Note that, by convention,  $\binom{j}{i} = 0$  for  $j < i$ .

We prove equation (3) by induction. When  $n = 0$ ,  $\mathbf{1}^{\otimes 0} = \mathbf{1} \stackrel{\text{def}}{=} \mathbf{1}_C \in C$ . Since  $\Phi$  is a  $C$ -algebra homomorphism, we have

$$\Phi(\mathbf{1}_C) = (\mathbf{1}, \mathbf{1}, \dots) = \left( \binom{k-1}{0} \mathbf{1} \right)_k.$$

This verifies equation (3) for  $n = 0$ . Assume that equation (3) is true for  $n$ . Then we have

$$\begin{aligned} \Phi(\mathbf{1}^{\otimes(n+1)}) &= \Phi(P_C(\mathbf{1}^{\otimes n})) \\ &= P'_C(\Phi(\mathbf{1}^{\otimes n})) \\ &= P'_C\left(\left(\binom{k-1}{n} \lambda^n \mathbf{1}\right)_k\right) \\ &= \lambda \left( \sum_{i=1}^{k-1} \binom{i-1}{n} \lambda^n \mathbf{1} \right)_k \\ &= \left( \binom{k-1}{n+1} \lambda^{n+1} \mathbf{1} \right)_k. \end{aligned}$$

This completes the induction and verifies the first equation in the proposition. The second equation follows from the first equation and Theorem 2.6. ■



### 3 Basic properties

We will first consider subalgebras, quotient algebras and colimits. Further properties of Baxter algebras will be studied in later sections.

#### 3.1 Subalgebras

**Proposition 3.1** *Let  $f : A \rightarrow B$  be an injective  $C$ -algebra homomorphism, and let  $A$  and  $B$  be flat as  $C$ -modules. Then the induced Baxter  $C$ -algebra homomorphisms  $\mathbb{I}\mathbb{I}_C(f) : \mathbb{I}\mathbb{I}_C(A) \rightarrow \mathbb{I}\mathbb{I}_C(B)$  and  $\widehat{\mathbb{I}\mathbb{I}}_C(f) : \widehat{\mathbb{I}\mathbb{I}}_C(A) \rightarrow \widehat{\mathbb{I}\mathbb{I}}_C(B)$  are injective.*

**Proof:** By the construction of  $\mathbb{I}\mathbb{I}_C(A) = \bigoplus_{n \in \mathbb{N}_+} A^{\otimes n}$ ,  $\mathbb{I}\mathbb{I}_C(f)$  is defined to be

$$\bigoplus_{n \in \mathbb{N}_+} f^{\otimes n} : \bigoplus_{n \in \mathbb{N}_+} A^{\otimes n} \rightarrow \bigoplus_{n \in \mathbb{N}_+} B^{\otimes n}$$

where  $f^{\otimes n} : A^{\otimes n} \rightarrow B^{\otimes n}$  is the tensor power of the  $C$ -module map  $f$ . Also by Theorem 2.3,  $\widehat{\mathbb{I}\mathbb{I}}_C(f)$  can be described as

$$\prod f^{\otimes n} : \prod_{n \in \mathbb{N}_+} A^{\otimes n} \rightarrow \prod_{n \in \mathbb{N}_+} B^{\otimes n}.$$

Thus we only need to prove that  $f^{\otimes n}$  is injective for all  $n \geq 1$ .  $f^{\otimes 1} = f$  is injective by assumption. Assume that  $f^{\otimes n}$  is injective. Since  $A$  is flat,  $A^{\otimes n}$  is also flat. So  $f : A \rightarrow B$  is injective implies that

$$\text{id}_{A^{\otimes n}} \otimes f : A^{\otimes(n+1)} = A^{\otimes n} \otimes A \rightarrow A^{\otimes n} \otimes B$$

is injective. By inductive assumption,  $f^{\otimes n} : A^{\otimes n} \rightarrow B^{\otimes n}$  is injective. Since  $B$  is flat,

$$f^{\otimes n} \otimes \text{id}_B : A^{\otimes n} \otimes B \rightarrow B^{\otimes n} \otimes B = B^{\otimes(n+1)}$$

is injective. Thus we have that

$$f^{\otimes(n+1)} = (\text{id}_{A^{\otimes n}} \otimes f) \circ (f^{\otimes n} \otimes \text{id}_B) : A^{\otimes(n+1)} \rightarrow B^{\otimes(n+1)}$$

is injective, finishing the induction. ■

#### 3.2 Baxter ideals

We now study Baxter ideals of  $\mathbb{I}\mathbb{I}_C(A)$  generated by ideals of  $A$ . Let  $I$  be an ideal of  $A$ . For each  $n \in \mathbb{N}$ , let  $I^{(n)}$  be the  $C$ -submodule of  $\mathbb{I}\mathbb{I}_C(A)$  generated by the subset  $\{\bigotimes_{i=0}^n x_i \mid x_i \in A, x_i \in I \text{ for some } 0 \leq i \leq n\}$ .

**Proposition 3.2** *Let  $I$  be an ideal of  $A$ . Let  $\tilde{I}$  be the Baxter ideal of  $\mathbb{I}\mathbb{I}_C(A)$  generated by  $I$  and let  $\hat{I}$  be the Baxter ideal of  $\hat{\mathbb{I}\mathbb{I}}_C(A)$  generated by  $I$ . Then*

$$\tilde{I} = \bigoplus_{k \in \mathbb{N}} I^{(n)} \subseteq \mathbb{I}\mathbb{I}_C(A)$$

and

$$\hat{I} = \prod_{k \in \mathbb{N}} I^{(n)} \subseteq \hat{\mathbb{I}\mathbb{I}}_C(A)$$

**Proof:** Denote

$$S = \{ \otimes_{i=0}^n x_i \mid x_i \in A, 0 \leq i \leq n, \text{ and } x_i \in I \text{ for some } 0 \leq i \leq n, n \in \mathbb{N} \}.$$

Then clearly

$$\bigoplus_{k \in \mathbb{N}} I^{(n)} = \sum_{x \in S} Cx.$$

So to prove  $\tilde{I} \subseteq \bigoplus_{k \in \mathbb{N}} I^{(n)}$ , we only need to prove

$$\tilde{I} \subseteq \sum_{x \in S} Cx.$$

Let  $J$  denote the sum on the right hand side. Since clearly  $I \subseteq J$ , we only need to prove that  $J$  is an Baxter ideal. Clearly  $J$  is a  $C$ -submodule of  $\mathbb{I}\mathbb{I}_C(A)$  and is closed under the Baxter operator  $P_A$ . For any  $x \in S$  and  $y = \otimes_{j=0}^m y_j \in A^{\otimes(m+1)}$ , we have

$$xy = x_0 y_0 \otimes \sum_{(\sigma, T) \in \bar{S}(n, m)} \lambda^{|T|} \sigma((\otimes_{i=1}^n x_i) \otimes (\otimes_{j=1}^m y_j); T).$$

From the definition of  $S$ , either  $x_0 \in I$  or  $x_i \in I$  for some  $1 \leq i \leq n$ . Thus in each term of the above sum, either  $x_0 y_0 \in I$  or one of the tensor factors of  $\sigma((\otimes_{i=1}^n x_i) \otimes (\otimes_{j=1}^m y_j))$  is in  $I$ . This shows that  $xy \in J$ . Thus  $J$  is an Baxter ideal of  $\mathbb{I}\mathbb{I}_C(A)$ . This proves  $\tilde{I} \subseteq \bigoplus_{k \in \mathbb{N}} I^{(n)}$ .

We next prove by induction on  $n$  that each  $I^{(n)}$  is in  $\tilde{I}$ . When  $n = 0$ , then  $x \in I^{(n)}$  means that  $x \in I$ . So the claim is true. Assuming that the claim is true for  $n$  and let  $x = \otimes_{i=0}^{n+1} x_i \in I^{(n+2)}$ . Then one of  $x_i$ ,  $0 \leq i \leq n+1$  is in  $I$ . If  $x_0 \in I$ , then  $x = x_0(1 \otimes x_1 \otimes \dots \otimes x_{n+1})$  is in  $\tilde{I}$  since  $\tilde{I}$  is the ideal of  $\mathbb{I}\mathbb{I}_C(A)$  generated by  $I$ . If  $x_i \in I$  for some  $1 \leq i \leq n+1$ , then in

$$x = x_0(1 \otimes x_1 \otimes \dots \otimes x_{n+1}) = x_0 P_A(x_1 \otimes \dots \otimes x_{n+1}),$$

$x_1 \otimes \dots \otimes x_{n+1} \in \tilde{I}$  by induction. Thus we again have  $x \in \tilde{I}$ . Since  $\tilde{I}$  is a  $C$ -submodule, we have  $I^{(n+1)} \subseteq \tilde{I}$ . This completes the induction. Therefore,  $\bigoplus_{n \in \mathbb{N}} I^{(n)} \subseteq \tilde{I}$ . This proves the first equation in the proposition.

To prove the second equation, note that by the construction of the isomorphism

$$\psi_A : \varprojlim (\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)) \rightarrow \prod_{k \in \mathbb{N}} A^{\otimes(k+1)}$$

in Theorem 2.3,

$$\begin{aligned} \prod_{k \in \mathbb{N}} I^{(k)} &\cong \varprojlim (\oplus_{k \in \mathbb{N}} I^{(k)} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A) \\ &= \varprojlim (\tilde{I} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A). \end{aligned}$$

So we only need to prove that

$$L \stackrel{\text{def}}{=} \varprojlim (\tilde{I} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$$

is the Baxter ideal  $\hat{I}'$  of  $\varprojlim (\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A))$  generated by  $I$ . For each  $n \in \mathbb{N}$ ,  $(\tilde{I} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$  is a Baxter ideal of  $\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$ . So the inverse limit  $\varprojlim (\tilde{I} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$  is a Baxter ideal of  $\varprojlim (\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A))$ . Therefore,  $\hat{I}' \subseteq L$ .

On the other hand, since  $\hat{I}'$  is a Baxter ideal of  $\varprojlim (\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A))$  containing  $I$ , its image  $\hat{I}'_n$  in  $\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$  is a Baxter ideal containing  $(I + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$ . By the same argument as in the proof of the first equation, we obtain that the Baxter ideal of  $\mathbb{I}\mathbb{I}_C(A)/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$  containing  $(I + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$  is

$$(\oplus_{k \leq n} I^{(k)} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A) = (\tilde{I} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A).$$

Therefore,  $\hat{I}'_n \supseteq (\tilde{I} + \text{Fil}^n \mathbb{I}\mathbb{I}_C(A))/\text{Fil}^n \mathbb{I}\mathbb{I}_C(A)$ . Taking the inverse limit, we obtain  $\hat{I}' \supseteq L$ , proving the second equation. ■

### 3.3 Quotient algebras

We can now describe how quotients are preserved under taking free Baxter algebras.

**Proposition 3.3** *Let  $I$  be an ideal of  $A$ . Let  $\tilde{I}$  be the Baxter ideal of  $\mathbb{I}\mathbb{I}_C(A)$  generated by  $I$  and let  $\hat{I}$  be the Baxter ideal of  $\hat{\mathbb{I}\mathbb{I}}_C(A)$  generated by  $I$ . Then*

$$\mathbb{I}\mathbb{I}_C(A/I) \cong \mathbb{I}\mathbb{I}_C(A)/\tilde{I}$$

and

$$\hat{\mathbb{I}\mathbb{I}}_C(A/I) \cong \hat{\mathbb{I}\mathbb{I}}_C(A)/\hat{I}$$

as Baxter  $C$ -algebras.

**Proof:** Let  $\pi : A \rightarrow A/I$  and  $\tilde{\pi} : \mathbb{I}\mathbb{I}_C(A) \rightarrow \mathbb{I}\mathbb{I}_C(A)/\tilde{I}$  be the natural surjections. The composite map

$$A \xrightarrow{j_A} \mathbb{I}\mathbb{I}_C(A) \xrightarrow{\tilde{\pi}} \mathbb{I}\mathbb{I}_C(A)/\tilde{I}$$

has kernel  $I$  by Proposition 3.2. Let  $j'_A : A/I \rightarrow \mathbb{I}\mathbb{I}_C(A)/\tilde{I}$  be the induced embedding. We only need to verify that  $\mathbb{I}\mathbb{I}_C(A)/\tilde{I}$  with the Baxter operator  $P'_A$  induced from  $P_A$ , and the embedding  $j'_A$  satisfies the universal property for a free Baxter  $C$ -algebra on  $A/I$ .

Let  $(R, P)$  be an Baxter  $C$ -algebra and let  $\varphi : A/I \rightarrow R$  be a  $C$ -algebra homomorphism. By the universal property of  $\mathbb{I}\mathbb{I}_C(A)$ , the  $C$ -algebra homomorphism

$$\eta \stackrel{\text{def}}{=} \varphi \circ \pi : A \rightarrow R$$

extends uniquely to an Baxter  $C$ -algebra homomorphism

$$\tilde{\eta} : (\mathbb{I}\mathbb{I}_C(A), P_A) \rightarrow (R, P).$$

Since  $I$  is in the kernel of  $\eta$ ,  $\tilde{I}$  is in the kernel of  $\tilde{\eta}$ , thus  $\tilde{\eta}$  induces uniquely an Baxter  $C$ -algebra homomorphism

$$\tilde{\eta}' : (\mathbb{I}\mathbb{I}_C(A), P'_A) \rightarrow (R, P).$$

We can summarize these maps in the following diagram

$$\begin{array}{ccccc}
 & A & \xrightarrow{j_A} & \mathbb{I}\mathbb{I}_C(A) & \\
 \pi \swarrow & & & & \searrow \tilde{\pi} \\
 A/I & \xrightarrow{j'_A} & \mathbb{I}\mathbb{I}_C(A)/\tilde{I} & & \\
 \varphi \searrow & \eta \swarrow & \tilde{\eta} \swarrow & \tilde{\eta}' \swarrow & \\
 & R & & & 
 \end{array}$$

We have

$$\begin{aligned}
 \varphi \circ \pi &= \eta \text{ (by definition)} \\
 &= \tilde{\eta} \circ j_A \text{ (by freeness of } \mathbb{I}\mathbb{I}_C(A) \text{ on } A) \\
 &= \tilde{\eta}' \circ \tilde{\pi} \circ j_A \text{ (by definition)} \\
 &= \tilde{\eta}' \circ j'_A \circ \pi \text{ (by definition)}.
 \end{aligned}$$

Since  $\pi$  is surjective, we have  $\varphi = \tilde{\eta}' \circ j'_A$ . If there is another  $\tilde{\eta}''$  such that  $\varphi = \tilde{\eta}'' \circ j'_A$ , then we have

$$\begin{aligned}
\tilde{\eta}' \circ j_A = \tilde{\eta}'' &\Rightarrow \tilde{\eta}' \circ j_A \circ \pi = \tilde{\eta}'' \circ \pi \\
&\Rightarrow \tilde{\eta}' \circ \tilde{\pi} \circ j_A = \tilde{\eta}'' \circ \tilde{\pi} \circ j_A \text{ ( by definition)} \\
&\Rightarrow \tilde{\eta}' \circ \tilde{\pi} = \tilde{\eta}'' \circ \tilde{\pi} \text{ ( by freeness of } \mathbb{I}\mathbb{I}\mathbb{I}_C(A) \text{ on } A) \\
&\Rightarrow \tilde{\eta}' = \tilde{\eta}'' \text{ ( by surjectivity of } \tilde{\pi}).
\end{aligned}$$

This proves the first equation of the proposition.

To prove the second equation, consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & \tilde{I} & \rightarrow & \mathbb{I}\mathbb{I}\mathbb{I}_C(A) & \xrightarrow{\mathbb{I}\mathbb{I}\mathbb{I}_C(\pi)} & \mathbb{I}\mathbb{I}\mathbb{I}_C(A/I) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \hat{I} & \rightarrow & \hat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(A) & \xrightarrow{\hat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(\pi)} & \hat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(A/I) \rightarrow 0
\end{array}$$

in which the vertical maps are injective. From the first part of the proposition, the top row is exact. The desired injectivity of the bottom row is clear, and the desired surjectivity follows from the definition of  $\hat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(\pi)$ . Also from the description of  $\hat{I}$  in Proposition 3.2,  $\hat{I} \subseteq \ker(\hat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(\pi))$ . On the other hand,

$$\begin{aligned}
(x_n)_n \in \ker(\hat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(\pi)) &\Leftrightarrow (\pi^{\otimes(n+1)}(x_n))_n = 0 \\
&\Leftrightarrow \pi^{\otimes(n+1)}(x_n) = 0, \forall n \geq 0 \\
&\Leftrightarrow x_n \in \ker(\pi^{\otimes(n+1)}), \forall n \geq 0 \\
&\Rightarrow x_n \in \ker(\mathbb{I}\mathbb{I}\mathbb{I}_C(\pi)), \forall n \geq 0 \\
&\Rightarrow x_n \in I^{(n)}, \forall n \geq 0 \\
&\Rightarrow (x_n) \in \hat{I}.
\end{aligned}$$

This proves the exactness of the bottom row, hence the second equation in the proposition. ■

### 3.4 Colimits

**Proposition 3.4** *Let  $\Lambda$  be a category whose objects form a set. Let  $F : \Lambda \rightarrow \mathbf{Alg}_C$  be a functor. Denote  $A_\lambda$  for  $F(\lambda)$ , and denote  $\text{colim}_\lambda$  for the colimit over  $\Lambda$ . Then  $\text{colim}_\lambda(\mathbb{I}\mathbb{I}\mathbb{I}_C(A_\lambda, P_{A_\lambda}))$  exists and*

$$\text{colim}_\lambda(\mathbb{I}\mathbb{I}\mathbb{I}_C(A_\lambda, P_{A_\lambda})) \cong (\mathbb{I}\mathbb{I}\mathbb{I}_C(\text{colim}_\lambda A_\lambda), P_{\text{colim}_\lambda A_\lambda}).$$

*In particular, for  $C$ -algebras  $A$  and  $B$ ,  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A \otimes B)$  is the coproduct of  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  and  $\mathbb{I}\mathbb{I}\mathbb{I}_C(B)$ .*

**Proof:** It is well-known that colimits exist in  $\mathbf{Alg}_C$ . The proposition then follows from the dual of [12, Theorem 1, p114], stated in page 115. ■

Similar statement for the complete free Baxter algebra is not true. For example, let  $\Lambda = \mathbb{N}_+$  and for each  $n \in \Lambda$ , let  $A_n = C[x_1, \dots, x_n]$ . With the natural inclusion,  $\{A_n\}$  is a direct system, with  $\text{colim}_n A_n = C[x_1, \dots, x_n, \dots]$ . We have  $\text{colim}_n (\widehat{\text{III}}_C(A_n), P_{A_n}) = \cup_n (\widehat{\text{III}}_C(A_n), P_{A_n})$  and  $\widehat{\text{III}}_C(\text{colim}_n A_n) = \widehat{\text{III}}_C(\cup_n A_n)$ . The element  $(\otimes_{i=1}^{k+1} x_i)_{k \in \mathbb{N}}$  is in  $\widehat{\text{III}}_C(\cup_n A_n)$ . But it is not in any  $\widehat{\text{III}}_C(A_n)$ , and hence is not in  $\cup_n (\widehat{\text{III}}_C(A_n), P_{A_n})$ .

## 4 Integral domains and reduced algebras

In this section, we investigate the question of when a free Baxter  $C$ -algebra or a free complete Baxter  $C$ -algebra is a domain and when it is a reduced algebra. We also study the nilpotent elements when the free Baxter algebra is not reduced. We will consider the case when  $C$  has characteristic zero in Section 4.1, and consider the case when  $C$  has positive characteristic in Section 4.2.

### 4.1 Case 1: $C$ has characteristic zero

We begin with the special case when  $C$  is a field. The general case will be reduced to this case.

#### 4.1.1 $\text{III}_C(A)$ and $\widehat{\text{III}}_C(A)$ when $C$ is a field

**Proposition 4.1** *Let  $C$  be a field of characteristic zero. Assume that  $A$  is a  $C$ -algebra and an integral domain.*

1.  $\text{III}_C(A)$  is an integral domain for any  $\lambda$ .
2.  $\widehat{\text{III}}_C(A)$  is an integral domain if and only if  $\lambda = 0$ .

**Proof:** 1. Let  $\Sigma$  be a basis set of  $A$  as a vector space over  $C$ , and let  $\prec$  be a linear order on  $\Sigma$ , assuming the axiom of choice. Thus

$$A = \bigoplus_{\mu \in \Sigma} C\mu$$

and consequently,

$$A^{\otimes n} = \bigoplus_{\mu \in \Sigma^n} C\mu$$

where  $\mu = (\mu_1, \dots, \mu_n) \in \Sigma^n$ . Let

$$\Sigma^\infty = \bigcup_{n \geq 1} \Sigma^n \text{ and } \bar{\Sigma}^\infty = \bigcup_{n \geq 0} \Sigma^n$$

with the convention that  $\Sigma^0$  is the singleton  $\{\phi\}$ . In the following, we identify a vector  $(\mu_1, \dots, \mu_n) \in \Sigma^n$  with the corresponding tensor  $\mu_1 \otimes \dots \otimes \mu_n \in A^{\otimes n}$ . Then as a  $C$ -vector space,

$$\mathbb{H}_C(A) = \bigoplus_{\mu \in \Sigma^\infty} C\mu.$$

It follows that, as a  $A$ -module,

$$\begin{aligned} \mathbb{H}_C(A) &= \bigoplus_{\mu \in \bar{\Sigma}^\infty} A \otimes \mu \\ &= \bigoplus_{\mu \in \bar{\Sigma}^\infty} A (\mathbf{1}_A \otimes \mu) \end{aligned}$$

with the convention that  $\mathbf{1}_A \otimes \phi = \mathbf{1}_A$ .

We next endow  $\bar{\Sigma}^\infty$  with the following variant of the lexicographic order induced from the order  $\prec$  on  $\Sigma$ . We define the empty set  $\phi$  to be the smallest element and, for  $\mu \in \Sigma^m$  and  $\nu \in \Sigma^n$ ,  $m, n > 0$ , define  $\mu \prec \nu$  if  $m < n$ , or  $m = n$  and for some  $1 \leq m_0 \leq m$  we have  $\mu_{m_0} \prec \nu_{m_0}$  and  $\mu_i = \nu_i$  for  $m_0 + 1 \leq i \leq m$ . We also denote this order on  $\bar{\Sigma}^\infty$  by  $\prec$ . It is a linear order. It is easy to check that, if  $\mu \prec \mu'$ , then for any  $\nu \in \bar{\Sigma}^\infty$ ,

$$\max\{\xi \mid \xi \in S(\mu, \nu)\} \prec \max\{\xi \mid \xi \in S(\mu', \nu)\}. \quad (4)$$

Here

$$S(\mu, \nu) = \{\sigma(\mu \otimes \nu) \mid \sigma \in S(m, n)\}$$

denotes the set of shuffles of  $\mu$  and  $\nu$ . Now let

$$x = \sum_{\mu \in \bar{\Sigma}^\infty} a_\mu (\mathbf{1}_A \otimes \mu), \quad a_\mu \in A,$$

and

$$y = \sum_{\nu \in \bar{\Sigma}^\infty} b_\nu (\mathbf{1}_A \otimes \nu), \quad b_\nu \in A$$

be two non-zero elements in  $\mathbb{H}_C(A)$ . When  $\lambda = 0$ , only admissible pairs  $(\sigma, T) \in \bar{S}(m, n)$  with empty  $T$  contribute to the mixable shuffle product defined in equation (2). So we have

$$\begin{aligned} xy &= \sum_{\mu, \nu \in \bar{\Sigma}^\infty} a_\mu b_\nu \sum_{\xi \in S(\mu, \nu)} \mathbf{1}_A \otimes \xi \\ &= \sum_{\xi \in \bar{\Sigma}^\infty} c_\xi (\mathbf{1}_A \otimes \xi). \end{aligned}$$

With these notations, we define

$$\mu_0 = \max\{\mu \mid a_\mu \neq 0\},$$

$$\nu_0 = \max\{\nu \mid b_\nu \neq 0\}$$

and

$$\xi_0 = \max\{\xi \mid c_\xi \neq 0\}.$$

Then from the inequality (4), we have

$$c_{\xi_0} = a_{\mu_0} b_{\nu_0} n_0$$

where  $n_0$  is the number of times that  $\xi_0$  occurs as a shuffle of  $\mu_0$  and  $\nu_0$ . If  $\lambda \neq 0$ , then there are extra terms in the equation (2) of  $xy$  that come from the mixable shuffles with admissible pairs in which  $T$  is non-empty. But these terms will have shorter lengths and hence are smaller in the order  $\prec$  than the terms from shuffles without any admissible pairs. So  $c_{\xi_0}$  given above is still the coefficient for the largest term. Note that  $n_0$  is a positive integer by definition. Since  $A$  is a domain, we have  $a_{\mu_0} b_{\nu_0} \neq 0$ . Since  $A$  has characteristic zero, we further have  $a_{\mu_0} b_{\nu_0} n_0 \neq 0$ . Since

$$xy = \sum_{\xi \in \overline{\Sigma}^\infty} c_\xi (\mathbf{1}_A \otimes \xi)$$

is the decomposition of  $xy$  according to the basis  $\overline{\Sigma}^\infty$  of the free  $A$ -module

$$\text{III}_C(A) = \sum_{\xi \in \overline{\Sigma}^\infty} A(\mathbf{1}_A \otimes \xi),$$

it follows that  $xy \neq 0$ .

2. Let  $\lambda = 0$ . The same proof as above, replacing max by min, shows that  $\widehat{\text{III}}_C(A)$  is an integral domain.

Let  $\lambda \neq 0$ . Consider  $x = \mathbf{1}_A^{\otimes 2}$ ,  $y = \sum_{n=0}^{\infty} (-\lambda)^{-n} \mathbf{1}_A^{\otimes(n+1)}$  in  $\widehat{\text{III}}_C(A)$ . By Proposition 2.2, we have

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} (-\lambda)^{-n} \mathbf{1}_A^{\otimes 2} \mathbf{1}_A^{\otimes(n+1)} \\ &= \sum_{n=0}^{\infty} (-\lambda)^{-n} \left( \binom{n+1}{n} \binom{n}{0} \lambda^0 \mathbf{1}_A^{\otimes(n+2)} + \binom{n}{n} \binom{n}{1} \lambda \mathbf{1}_A^{\otimes(n+1)} \right) \\ &= \sum_{n=0}^{\infty} (-\lambda)^{-n} (n+1) \mathbf{1}_A^{\otimes(n+2)} + \sum_{n=0}^{\infty} (-\lambda)^{-n} n \lambda \mathbf{1}_A^{\otimes(n+1)} \\ &= \sum_{n=0}^{\infty} (-\lambda)^{-n} (n+1) \mathbf{1}_A^{\otimes(n+2)} - \sum_{n=1}^{\infty} (-\lambda)^{-n+1} n \mathbf{1}_A^{\otimes(n+1)} \\ &= 0. \end{aligned}$$



So  $\widehat{\text{III}}_C(A)$  is not an integral domain. ■

#### 4.1.2 $\text{III}_C(A)$ for a general ring $C$

Now let  $C$  be any ring. For a  $C$ -module  $N$ , denote

$$N_{\text{tor}} = \{x \in N \mid rx = 0 \text{ for some } r \in C, r \neq 0\}$$

for the  $C$ -torsion submodule of  $N$ . For a domain  $D$ , denote  $\text{Fr}(D)$  for the quotient field of  $D$ .

**Theorem 4.2** *Let  $A$  be a  $C$ -algebra of characteristic zero, with the  $C$ -algebra structure given by  $\varphi : C \rightarrow A$ . Denote  $I_0 = \ker \varphi$ . The following statements are equivalent.*

1.  $\text{III}_C(A)$  is a domain.
2.  $A$  is a domain and  $(A^{\otimes n})_{\text{tor}} = I_0$ , for all  $n \geq 1$ .
3.  $A$  is a domain and the natural map  $A^{\otimes n} \rightarrow \text{Fr}(C/I_0) \otimes A^{\otimes n}$  is injective for all  $n \geq 1$ .

**Proof:** Let  $\bar{C} = C/I_0$ . Then  $A$  is also a  $\bar{C}$ -algebra. It is well-known that the tensor product  $A \otimes_C A$  is canonically isomorphic to  $A \otimes_{\bar{C}} A$  as  $C$ -modules and as  $\bar{C}$ -modules. It follows that, as a ring, the  $C$ -algebra  $\text{III}_C(A)$  is canonically isomorphic to the  $\bar{C}$ -algebra  $\text{III}_{\bar{C}}(A)$ . Since being an integral domain is a property of a ring,  $\text{III}_C(A)$  is a domain if and only if  $\text{III}_{\bar{C}}(A)$  is one. Similarly,  $\widehat{\text{III}}_C(A)$  is a domain if and only if  $\widehat{\text{III}}_{\bar{C}}(A)$  is one. Thus we only need prove the theorem in the case when  $\varphi : C \rightarrow A$  is injective. So we can assume that  $I_0 = 0$ . We will make this assumption for the rest of the proof.

First note that if  $A$  is a domain, then  $C$  is also a domain. In this case we denote  $S = C - \{0\}$  and  $F = \text{Fr}(C)$ .

(2  $\Leftrightarrow$  3). This follows from the fact [1, Exercise 3.12] that, for each  $n \geq 1$ ,

$$(A^{\otimes n})_{\text{tor}} = \ker\{A^{\otimes n} \rightarrow F \otimes_C A^{\otimes n}\}.$$

Therefore the second and the third statement are equivalent.

(3  $\Rightarrow$  1). We have the natural isomorphisms  $F \cong S^{-1}C$ ,  $F \otimes A \cong S^{-1}A$  and

$$S^{-1}(A^{\otimes n}) \cong (S^{-1}A)_F^{\otimes n} \stackrel{\text{def}}{=} \underbrace{S^{-1}A \otimes_F \dots \otimes_F S^{-1}A}_{n\text{-factors}} \cong (S^{-1}A)^{\otimes n}. \quad (5)$$

Here the first isomorphism is from [1, Proposition 3.3.7] and the last isomorphism follows from the definition of tensor products and the assumption that  $C$  and  $A$  are domains. By the universal property of  $\text{III}_C(A)$  as a

free Baxter algebra, the natural  $C$ -algebra homomorphism  $f : A \rightarrow S^{-1}A$  gives a  $C$ -algebra homomorphism  $\text{III}_C(f) : \text{III}_C(A) \rightarrow \text{III}_C(S^{-1}A)$ . In fact,  $\text{III}_C(f) = \bigoplus_{n=1}^{\infty} f^{\otimes n}$  where  $f^{\otimes n}$  is the tensor power of  $f$ . By equation (5),

$$f^{\otimes n} : A^{\otimes n} \rightarrow (S^{-1}A)^{\otimes n} \cong (S^{-1}A)_F^{\otimes n} \cong F \otimes (A^{\otimes n}).$$

Thus we have a  $C$ -algebra homomorphism  $\tilde{f} : \text{III}_C(A) \rightarrow \text{III}_F(S^{-1}A)$  and, by the third statement of the proposition,  $f^{\otimes n}$  is injective. Therefore  $\text{III}_C(A)$  is identified with a  $C$  subalgebra of  $\text{III}_F(S^{-1}A)$  via  $\tilde{f}$ , and hence is a domain since  $\text{III}_F(S^{-1}A)$  is a domain by Proposition 4.1.

(1  $\Rightarrow$  2). If  $\text{III}_C(A)$  is a domain, then its subring  $A$  is a domain. Since  $C$  is a subring of  $A$  and hence of  $\text{III}_C(A)$ , we have  $\text{III}_C(A)_{\text{tor}} = 0$ . Since  $\text{III}_C(A) = \bigoplus_{n \in \mathbb{N}_+} A^{\otimes n}$ ,  $(A^{\otimes n})_{\text{tor}} = 0$  for all  $n \in \mathbb{N}_+$ . ■

**Corollary 4.3** *Let  $C$  be a domain of characteristic zero.*

1. *If  $A$  is a flat  $C$ -algebra, i.e.,  $A$  is a  $C$ -algebra and is flat as a  $C$ -module, then  $\text{III}_C(A)$  is a domain. In particular, for any set  $X$ ,  $\text{III}_C(X)$  is a domain.*
2. *If  $C$  is a Dedekind domain, then for a  $C$ -algebra  $A$ , the free Baxter algebra  $\text{III}_C(A)$  is a domain if and only if  $A$  is torsion free.*

**Proof:** As in the proof of Theorem 4.2, we can assume that  $C$  is a subring of  $A$ .

1. If  $A$  is a flat  $C$ -module, then  $A^{\otimes n}$ ,  $n \geq 1$  are flat  $C$ -modules, so from the injective map  $C \rightarrow F$  of  $C$ -modules, we obtain the injective map  $A^{\otimes n} = C \otimes A^{\otimes n} \rightarrow F \otimes A^{\otimes n}$ . Hence by Theorem 4.2,  $\text{III}_C(A)$  is a domain.

2. If  $C$  is a Dedekind domain, then  $A$  is a flat  $C$ -module if and only if  $A$  is torsion free. Hence the statement. ■

**Corollary 4.4** *Let  $A$  be a  $C$  algebra given by the ring homomorphism  $\varphi : C \rightarrow A$ . Let  $I$  be a prime ideal of  $A$ . The Baxter ideal  $\tilde{I}$  (see Proposition 3.2) of  $\text{III}_C(A)$  generated by  $I$  is a prime ideal if and only if  $A/I$  has characteristic zero and  $((A/I)^{\otimes n})_{\text{tor}} = \ker \varphi$  for all  $n \geq 1$ .*

**Proof:** By Proposition 3.3,  $\tilde{I}$  is a prime ideal if and only if  $\text{III}_C(A/I)$  is a domain. If  $A/I$  has characteristic zero and  $((A/I)^{\otimes n})_{\text{tor}} = \ker \varphi$  for all  $n \geq 1$ , then by Theorem 4.2,  $\text{III}_C(A/I)$  is a domain. Conversely, if  $A/I$  has non-zero characteristic, then by Theorem 4.8 (the proof of which is independent of Theorem 4.2),  $\text{III}_C(A/I)$  is not a domain. If  $A/I$  has zero characteristic, but  $((A/I)^{\otimes n})_{\text{tor}} \neq \ker \varphi$  for some  $n \geq 1$ , then by Theorem 4.2,  $\text{III}_C(A/I)$  is not a domain. This proves the corollary. ■

### 4.1.3 $\widehat{\text{III}}_C(A)$ for a general ring $C$

We now consider complete Baxter algebras.

**Lemma 4.5** *Let  $C$  be a UFD and let  $x \in \widehat{\text{III}}_C(X)$  be non-zero.*

1. *Let  $\lambda \in C$  be a prime element. There is  $m \in \mathbb{N}$  such that  $x = \lambda^m x'$  and such that  $x' \notin \lambda \widehat{\text{III}}_C(X)$ .*
2. *Let  $\lambda \in C$  be non-zero. If  $\lambda x = 0$ , then  $x = 0$ .*

**Proof:** 1. By Theorem 2.3, any element  $x \in \widehat{\text{III}}_C(X)$  has a unique expression of the form

$$x = \sum_{n=0}^{\infty} x_n, \quad x_n \in \text{III}_C^n(X) = C[X]^{\otimes(n+1)}.$$

So  $x \neq 0$  if and only if  $x_{n_0} \neq 0$  for some  $n_0 \in \mathbb{N}$ . Let  $M(X)$  be the free commutative monoid on  $X$ . Define

$$\overline{X}_n = \{\otimes_{i=1}^n u_i \mid u_i \in M, 1 \leq i \leq n\}.$$

Then we have

$$C[X]^{\otimes(n+1)} = \oplus_{u \in \overline{X}_{n+1}} Cu$$

and  $x_{n_0}$  can be uniquely expressed as  $x_{n_0} = \sum_{u \in \overline{X}_{n_0+1}} c_u u$ . Thus  $x_{n_0} \neq 0$  implies that  $c_{u_0} \neq 0$  for some  $u_0 \in \overline{X}_{n_0+1}$ . Since  $C$  is a UFD, there is  $m_0 \in \mathbb{N}_+$  such that  $c_{u_0} \notin \lambda^{m_0} C$ . Then  $x_{n_0} \notin \lambda^{m_0} C[X]^{\otimes(n_0+1)}$  and  $x \notin \lambda^{m_0} \widehat{\text{III}}_C(X)$ . Therefore the integer

$$\max\{k \mid k \in \mathbb{N}, x \in \lambda^k \widehat{\text{III}}_C(X)\}$$

exists. This integer can be taken to be the  $m$  in the first statement of the lemma.

2. Assume that  $x \in \widehat{\text{III}}_C(X)$  is non-zero. Then as in the proof of the first part of the lemma, there is  $n_0 \in \mathbb{N}$  such that

$$x = \sum_{n=0}^{\infty} x_n, \quad x_n \in \text{III}_C^n(X) = C[X]^{\otimes(n+1)}$$

and  $x_{n_0} \neq 0$ . Also, there is  $u_0 \in \overline{X}_{n_0+1}$  such that

$$x_{n_0} = \sum_{u \in \overline{X}_{n_0+1}} c_u u, \quad c_u \in C$$

and  $c_{u_0} \neq 0$ . Since  $C$  is a domain and  $\lambda \neq 0$ , we have  $\lambda c_{u_0} \neq 0$ . Since  $C[X]^{\otimes(n_0+1)}$  is a free  $C$ -module with the set  $\overline{X}_{n_0+1}$  as a basis, we have  $\lambda x_{n_0+1} \neq 0$ . This in turn proves  $\lambda x \neq 0$ . ■

**Theorem 4.6** *Let  $C$  be a  $\mathbb{Q}$ -algebra and a domain with the property that for every maximal ideal  $M$  of  $C$ , the localization  $C_M$  of  $C$  at  $M$  is a UFD. Let  $X$  be a set. For  $\lambda \in C$ ,  $\widehat{\text{III}}_C(X)$  is a domain if and only if  $\lambda$  is not a unit.*

**Remarks:** 1. If  $C$  is the affine ring of a nonsingular affine variety on a field of characteristic zero, then  $C$  is locally factorial [7, p. 257]. Hence Theorem 4.6 applies.

2. If  $C$  is not a  $\mathbb{Q}$ -algebra, the statement in the theorem does not hold. See the example after the proof.

**Proof:** Assume that  $\lambda$  is a unit. Consider the elements  $x = \mathbf{1}_C^{\otimes 2}$ ,  $y = \sum_{n=0}^{\infty} (-\lambda)^{-n} \mathbf{1}_C^{\otimes (n+1)}$  in  $\widehat{\text{III}}_C(X)$ . As in the proof of Proposition 4.1, we verify that  $xy = 0$ . So  $\widehat{\text{III}}_C(X)$  has zero divisors and is not a domain.

Now assume that  $\lambda \in C$  is not a unit. We will prove that  $\widehat{\text{III}}_C(X)$  is a domain. We will carry out the proof in four steps.

**Step 1:** We first assume that  $C$  is a  $\mathbb{Q}$ -algebra and a domain, and assume that  $\lambda \in C$  is zero. We do not assume that, for every maximal ideal  $M$  of  $C$ ,  $C_M$  is a UFD. We clearly have  $(C[X]^{\otimes n})_{\text{tor}} = 0$ . Thus from the proof of  $2 \Leftrightarrow 3$  and  $3 \Rightarrow 1$  in Theorem 4.2, the natural map

$$C[X]^{\otimes n} \rightarrow (\text{Fr}(C)[X])_{\text{Fr}(C)}^{\otimes n}$$

is injective. So

$$\widehat{\text{III}}_C(X) \rightarrow \widehat{\text{III}}_{\text{Fr}(C)}(X)$$

is injective. By Proposition 4.1,  $\widehat{\text{III}}_{\text{Fr}(C)}(X)$  is a domain. Therefore,  $\widehat{\text{III}}_C(X)$  is a domain.

**Step 2:** Next assume that  $C$  is a UFD and  $\lambda \in C$  is a prime element. Then the ideal  $\lambda C$  of  $C$  is a prime ideal. Hence  $C/\lambda C$  is a domain. Then  $C[X]/\lambda C[X] \cong (C/\lambda C)[X]$  is also a domain. Note that

$$\begin{aligned} \widehat{\text{III}}_C(C[X]/\lambda C[X]) &\cong \widehat{\text{III}}_{C/\lambda C}(C[X]/\lambda C[X]) \\ &\cong \widehat{\text{III}}_{C/\lambda C}((C/\lambda C)[X]) \\ &= \widehat{\text{III}}_{C/\lambda C}(X) \end{aligned}$$

as Baxter  $C$ -algebras.  $C/\lambda C$  is a  $\mathbb{Q}$ -algebra and a domain. So from the first step of the proof, the weight 0 Baxter  $C/\lambda C$ -algebra  $\widehat{\text{III}}_{C/\lambda C}(X)$  is a domain. So  $\widehat{\text{III}}_C(C[X]/\lambda C[X])$  is a domain. On the other hand, by Proposition 3.2,  $\lambda \widehat{\text{III}}_C(X)$  is the Baxter ideal of  $\widehat{\text{III}}_C(X)$  generated by  $\lambda C[X]$ . So by Proposition 3.3,

$$\widehat{\text{III}}_C(C[X]/\lambda C[X]) \cong \widehat{\text{III}}_C(X)/\lambda \widehat{\text{III}}_C(X).$$

Thus  $\lambda\widehat{\mathbb{M}}_C(X)$  is a prime ideal.

Suppose  $\widehat{\mathbb{M}}_C(X)$  is not a domain. Then there are non-zero elements  $x, y \in \widehat{\mathbb{M}}_C(X)$  such that  $xy = 0$ . From the first statement of Lemma 4.5, there are  $m, n \in \mathbb{N}$  such that  $x = \lambda^m x', y = \lambda^n y'$  and  $x', y' \notin \lambda\widehat{\mathbb{M}}_C(X)$ . From the second statement of Lemma 4.5, we have

$$xy = 0 \Rightarrow \lambda^{m+n} x' y' = 0 \Rightarrow x' y' = 0.$$

In particular, we have  $x' y' \in \lambda\widehat{\mathbb{M}}_C(X)$ . Since  $\lambda\widehat{\mathbb{M}}_C(X)$  is a prime ideal, we must have  $x' \in \lambda\widehat{\mathbb{M}}_C(X)$  or  $y' \in \lambda\widehat{\mathbb{M}}_C(X)$ . This is contradiction.

**Step 3:** We next assume that the  $\mathbb{Q}$ -algebra  $C$  is a domain and a UFD, and assume that  $\lambda \in C$  is not a unit. Then there is a prime element  $\lambda_1 \in C$  such that  $\lambda = \lambda_1 \lambda_2$  for some  $\lambda_2 \in C$ . From the second step of the proof, the weight  $\lambda_1$  complete shuffle algebra  $(\widehat{\mathbb{M}}_{C, \lambda_1}(X), P_{X, \lambda_1})$  is a domain.

Define another operator  $Q$  on the  $C$ -algebra  $\widehat{\mathbb{M}}_{C, \lambda_1}(X)$  by

$$Q(x) = \lambda_2 P_{X, \lambda_1}(x), \quad x \in \widehat{\mathbb{M}}_{C, \lambda_1}(X).$$

Then

$$\begin{aligned} Q(x)Q(y) &= \lambda_2^2 P_{X, \lambda_1}(x) P_{X, \lambda_1}(y) \\ &= \lambda_2^2 (P_{X, \lambda_1}(x P_{X, \lambda_1}(y)) + P_{X, \lambda_1}(y P_{X, \lambda_1}(x)) + \lambda_1 P_{X, \lambda_1}(xy)) \\ &= \lambda_2 P_{X, \lambda_1}(x \lambda_2 P_{X, \lambda_1}(y)) + \lambda_2 P_{X, \lambda_1}(y \lambda_2 P_{X, \lambda_1}(x)) + \lambda \lambda_2 P_{X, \lambda_1}(xy) \\ &= Q(xQ(y)) + Q(yQ(x)) + \lambda Q(xy) \end{aligned}$$

So  $(\widehat{\mathbb{M}}_{C, \lambda_1}(X), Q)$  is a Baxter algebra of weight  $\lambda$ . Since  $(\widehat{\mathbb{M}}_{C, \lambda}(X), P_{X, \lambda})$  is a free complete Baxter algebra of weight  $\lambda$ , there is a unique homomorphism of weight  $\lambda$  complete Baxter algebras

$$f : (\widehat{\mathbb{M}}_{C, \lambda}(X), P_{X, \lambda}) \rightarrow (\widehat{\mathbb{M}}_{C, \lambda_1}(X), Q)$$

that extends the identity map on  $X$ .

**Lemma 4.7** *The map  $f$  is injective.*

**Proof:** We first prove that, for  $n \in \mathbb{N}$  and  $x \in \mathbb{M}_{C, \lambda}^n(X)$ ,

$$f(x) = \lambda_2^n x \in \mathbb{M}_{C, \lambda_1}^n(X). \quad (6)$$

Recall that, as a  $C$ -module,  $\mathbb{M}_C^n(X) = C[X]^{\otimes(n+1)}$ . The identify map on  $X$  induces the identity map on  $C[X]$ . This proves equation (6) for  $n = 0$ .

Assuming that, for  $x \in \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda}^n(X)$ ,  $f(x) = \lambda_2^n x \in \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda_1}^n(X)$ , and letting  $x_0 \otimes \dots \otimes x_{n+1} \in \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda}^{n+1}(X)$ , we have

$$\begin{aligned}
f(x) &= f(x_0 P_{X,\lambda}(x_1 \otimes \dots \otimes x_{n+1})) \\
&= f(x_0) f(P_{X,\lambda}(x_1 \otimes \dots \otimes x_{n+1})) \\
&= x_0 Q(f(x_1 \otimes \dots \otimes x_{n+1})) \\
&= x_0 Q(\lambda_2^n x_1 \otimes \dots \otimes x_{n+1}) \\
&= x_0 \lambda_2 P_{X,\lambda_1}(\lambda_2^n x_1 \otimes \dots \otimes x_{n+1}) \\
&= \lambda_2^{n+1} x_0 \otimes x_1 \otimes \dots \otimes x_{n+1}.
\end{aligned}$$

This implies that, for any  $y \in \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda}^{n+1}(X)$ , we have  $f(y) = \lambda^{n+1} y \in \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda_1}^{n+1}(X)$ . This completes the induction.

By equation (6), the restriction of  $f$  to  $\mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda}^n(X)$  gives a map

$$f_n : \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda}^n(X) \rightarrow \mathbb{I}\mathbb{I}\mathbb{I}_{C,\lambda_1}^n(X)$$

and, from Theorem 2.3, we have

$$f = \prod_{n \in \mathbb{N}} f_n.$$

Thus in order to prove the lemma, we only need to prove that  $f_n$  is injective for each  $n \in \mathbb{N}$ . This follows from equation (6) and Lemma 4.5. ■

We continue the proof of Theorem 4.2. Since  $\lambda_1$  is a prime element of  $C$ , by the first part of the proof,  $\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_{C,\lambda_1}(X)$  is a domain. By Lemma 4.7,  $\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_{C,\lambda}(X)$  is isomorphic to a subalgebra of  $\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_{C,\lambda_1}(X)$ , hence is also a domain.

**Step 4:** We now consider the general case when the  $\mathbb{Q}$ -algebra  $C$  is a domain whose localization at each maximal ideal is a UFD. Let  $\lambda$  be a non-unit. Then there is a maximal ideal  $M$  of  $C$  containing  $\lambda$ . By assumption,  $C_M$  is a UFD. If we regard  $C$  as a subring of  $C_M$  by the natural embedding  $C \rightarrow C_M$  given by the localization map, then  $\lambda \in C$  remains a non-unit in  $C_M$ . From the proof in Step 3,  $\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_{C_M,\lambda}(X)$  is domain. On the other hand, the natural embeddings

$$C[X] \hookrightarrow C_M[X] \hookrightarrow \text{Fr}(C)[X]$$

induce the natural morphisms of  $C$ -modules.

$$C[X]^{\otimes n} \rightarrow C_M[X]^{\otimes n} \rightarrow \text{Fr}(C)[X]^{\otimes n}.$$

Since clearly  $(A^{\otimes n})_{\text{tor}} = 0$  for  $A = C[X]$ , by the equivalence (2)  $\Leftrightarrow$  (3) in Theorem 4.2, the composite of the above two maps is injective. Therefore

the map  $C[X]^{\otimes n} \rightarrow C_M[X]^{\otimes n}$  is injective. Thus  $\widehat{\mathbb{M}}_{C,\lambda}(X) \rightarrow \widehat{\mathbb{M}}_{C_M,\lambda}(X)$  is injective. This proves that  $\widehat{\mathbb{M}}_{C,\lambda}(X)$  is a domain. ■

When  $C$  is not a  $\mathbb{Q}$ -algebra, the situation is much more complicated and will be the subject of a further study. Here we just give an example to show that Theorem 4.6 does not hold without the assumption that  $C$  is a  $\mathbb{Q}$ -algebra.

**Example:** Let  $C = \mathbb{Z}$  and  $\lambda = 2$ . Then  $C$  is not a  $\mathbb{Q}$ -algebra. But all other conditions in Theorem 4.6 are satisfied. Consider the two elements

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \mathbf{1}^{\otimes(n+1)}$$

and

$$y = 2 + \sum_{n=1}^{\infty} (-1)^n \mathbf{1}^{\otimes(n+1)}.$$

in  $\widehat{\mathbb{M}}_C(X) = \widehat{\mathbb{M}}_C(C[X])$ . By Proposition 2.7, the  $n$ -th component of  $\Phi(x)$  is

$$\begin{aligned} \sum_{i=1}^n \binom{n}{i} (-1)^{i+1} 2^i &= - \sum_{i=1}^n \binom{n}{i} (-2)^i \\ &= -((1-2)^n - 1) \\ &= \begin{cases} 0, & n \text{ even}, \\ 2, & n \text{ odd} \end{cases} \end{aligned}$$

and the  $n$ -th component of  $\Phi(y)$  is

$$\begin{aligned} 2 + \sum_{i=1}^n \binom{n}{i} (-1)^i 2^i &= 2 + \sum_{i=1}^n \binom{n}{i} (-2)^i \\ &= 2 + ((1-2)^n - 1) \\ &= \begin{cases} 2, & n \text{ even}, \\ 0, & n \text{ odd} \end{cases} \end{aligned}$$

Thus  $\Phi(xy) = \Phi(x)\Phi(y) = 0$ . Therefore,  $xy = 0$  by Theorem 2.4. This shows that  $\widehat{\mathbb{M}}_C(X)$  is not a domain.

## 4.2 Case 2: $C$ has positive characteristic

We now consider the case when the characteristic of  $C$  is positive.

**Theorem 4.8** *Let  $C$  be a ring of positive characteristic and let  $A \supseteq C$  be a  $C$ -algebra.*

1.  $\mathbb{I}\mathbb{I}_C(A)$  (hence  $\widehat{\mathbb{I}\mathbb{I}}_C(A)$ ) is not an integral domain.
2. If  $\lambda = 0$ , then  $\mathbb{I}\mathbb{I}_C(A)$  (hence  $\widehat{\mathbb{I}\mathbb{I}}_C(A)$ ) is not reduced. The nil radical of  $\mathbb{I}\mathbb{I}_C(A)$  (resp. of  $\widehat{\mathbb{I}\mathbb{I}}_C(A)$ ) is given by

$$N(\mathbb{I}\mathbb{I}_C(A)) = N(A) \oplus \left( \bigoplus_{n \in \mathbb{N}_+} A^{\otimes(n+1)} \right)$$

$$(\text{resp. } N(\widehat{\mathbb{I}\mathbb{I}}_C(A)) \subseteq N(A) \times \prod_{n \in \mathbb{N}_+} A^{\otimes(n+1)}).$$

3. If  $\lambda \neq 0$ , and if, for every  $k \geq 1$ ,  $\lambda$  has trivial annihilator in the  $C$ -module  $A^{\otimes k}$  and the tensor power algebra  $A^{\otimes k}$  is reduced, then  $\mathbb{I}\mathbb{I}_C(A)$  and  $\widehat{\mathbb{I}\mathbb{I}}_C(A)$  are reduced.

**Proof:** 1. Since  $A$  is a subalgebra of  $\mathbb{I}\mathbb{I}_C(A)$ , it is clear that if  $A$  is not a domain, then  $\mathbb{I}\mathbb{I}_C(A)$  is not a domain. So we will assume that  $A$  is a domain, hence the characteristic of  $A$  is a prime number  $p$ .

If  $\lambda = 0$ , then from Proposition 2.2,  $(\mathbf{1} \otimes \mathbf{1})^p = (p!) \mathbf{1}^{\otimes p} = 0$  in  $\mathbb{I}\mathbb{I}_C(C)$ . Then  $(\mathbf{1}_A \otimes \mathbf{1}_A)^p = (p!) \mathbf{1}_A^{\otimes(p+1)} = 0$  in  $\mathbb{I}\mathbb{I}_C(A)$ . So  $\mathbf{1}_A \otimes \mathbf{1}_A$  is a zero divisor and  $\mathbb{I}\mathbb{I}_C(A)$  is not a domain.

We now assume  $\lambda \neq 0$ . We first let  $A = C$ . Then by Proposition 2.7 the isomorphism of Baxter algebras

$$\Phi : \mathbb{I}\mathbb{I}_C(C) \rightarrow \mathfrak{S}(C) \subseteq \prod_{k \in \mathbb{N}_+} C$$

sends  $\mathbf{1}$  to  $(\mathbf{1}, \mathbf{1}, \dots)$  and sends  $\mathbf{1}^{\otimes 2} = P_C(\mathbf{1})$  to

$$P'_C(\mathbf{1}, \mathbf{1}, \dots) = \lambda(0, \mathbf{1}, 2\mathbf{1}, \dots).$$

Thus for each  $0 \leq i \leq p-1$ , we have

$$\Phi(i\lambda\mathbf{1} + \mathbf{1}^{\otimes 2}) = \lambda(i\mathbf{1}, (1+i)\mathbf{1}, (2+i)\mathbf{1}, \dots)$$

So the  $n$ -th component of  $\Phi(i\lambda\mathbf{1} + \mathbf{1}^{\otimes 2})$  is zero for  $n \equiv i \pmod{p}$ . Since the product in  $\mathfrak{S}(C)$  is componentwise, it follows that the  $n$ -th component of  $\prod_{i=0}^{p-1} \Phi(i\lambda\mathbf{1} + \mathbf{1}^{\otimes 2})$  is zero for all  $n$ . Therefore  $\prod_{i=0}^{p-1} \Phi(i\lambda\mathbf{1} + \mathbf{1}^{\otimes 2}) = 0$ . Since  $\Phi$  is an algebra isomorphism, we have

$$\prod_{i=0}^{p-1} (i\lambda\mathbf{1} + \mathbf{1}^{\otimes 2}) = 0.$$

Clearly none of  $i\lambda\mathbf{1} + \mathbf{1}^{\otimes 2}$ ,  $0 \leq i \leq p-1$ , is zero. So these elements are zero divisors and  $\mathbb{I}\mathbb{I}_C(C)$  is not a domain. For a general  $C$ -algebra  $A$  of



characteristic  $p$ , the ring homomorphism  $\varphi : C \rightarrow A$  that defines the  $C$ -algebra structure on  $A$  induces a Baxter algebra homomorphism  $\text{III}_C(\varphi) : \text{III}_C(C) \rightarrow \text{III}_C(A)$  sending  $\mathbf{1}^{\otimes k}$  to  $\mathbf{1}_A^{\otimes k}$ . So  $\text{III}_C(\varphi)$  sends

$$\prod_{i=0}^{p-1} (i\lambda \mathbf{1} + \mathbf{1}^{\otimes 2}) = 0$$

to

$$\prod_{i=0}^{p-1} (i\lambda \mathbf{1}_A + \mathbf{1}_A^{\otimes 2}) = 0.$$

This shows that  $\text{III}_C(A)$  is not a domain.

2. Let  $q > 0$  be the characteristic of  $C$ . If  $\lambda = 0$ , then as in the proof of the first statement of the theorem,  $(\mathbf{1}_A \otimes \mathbf{1}_A)^q = q! \mathbf{1}_A^{\otimes(q+1)} = 0$  in  $\text{III}_C(A)$ . So  $\mathbf{1}_A \otimes \mathbf{1}_A$  is a nilpotent and  $\text{III}_C(A)$  is not reduced.

Before describing the nil radical, we need some preparation. Let  $(R, P)$  be a Baxter algebra. For any  $x \in R$ , denote  $P_x(y) = P(xy)$ ,  $y \in R$ . For any  $n \in \mathbb{N}$ , denote  $P_x^n = \underbrace{P_x \circ \dots \circ P_x}_{n\text{-times}}$  with the convention that  $P_x^0 = \text{id}_R$ .

**Lemma 4.9** *Let  $(R, P)$  be a Baxter  $C$ -algebra of weight zero.*

1. For  $n \in \mathbb{N}$ ,  $P_x^n(\mathbf{1}_R)P_x(\mathbf{1}_R) = (n+1)P_x^{n+1}(\mathbf{1}_R)$ .
2. For  $n \in \mathbb{N}$ ,  $P(x)^n = n!P_x^n(\mathbf{1}_R)$ .

**Proof:** We prove both statements by induction on  $n$ . The first statement is clearly true for  $n = 0$ . Assume that it is true for  $n$ . Then

$$\begin{aligned} P_x^{n+1}(\mathbf{1}_R)P_x(\mathbf{1}_R) &= P(xP_x^n(\mathbf{1}_R))P(x) \\ &= P(xP(xP_x^n(\mathbf{1}_R))) + P(xP_x^n(\mathbf{1}_R)P(x)) \\ &= P_x^{n+2}(\mathbf{1}_R) + P(x(n+1)P_x^{n+1}(\mathbf{1}_R)) \\ &= (n+2)P_x^{n+2}(\mathbf{1}_R), \end{aligned}$$

completing the induction.

The second statement is again clear for  $n = 0$ . Assume that the statement is true for  $n$ . Then by the first statement of the lemma,

$$P(x)^{n+1} = P(x)^n P(x) = n!P_x^n(\mathbf{1}_R)P_x(\mathbf{1}_R) = (n+1)!P_x^{n+1}(\mathbf{1}_R),$$

completing the induction. ■

Now back to the proof of Theorem 4.8. We first prove

$$N(\mathbb{I}\mathbb{I}\mathbb{I}_C(A)) \supseteq N(A) \oplus \left( \bigoplus_{n \in \mathbb{N}_+} A^{\otimes(n+1)} \right).$$

Clearly  $N(A) \subseteq N(\mathbb{I}\mathbb{I}\mathbb{I}_C(A))$ . Let  $n \in \mathbb{N}_+$  and let  $x = x_0 \otimes \dots \otimes x_n \in A^{\otimes(n+1)}$ . Denote  $x = x_0 \otimes x^+$ , with  $x^+ = x_1 \otimes \dots \otimes x_n$ . We have  $x = x_0 \diamond P(x^+)$ . Then from Lemma 4.9,

$$x^q = x_0^q \diamond P(x^+)^q = x_0^q \diamond (q! P_{x^+}^q(\mathbf{1}_A)) = 0.$$

Since any element of  $\bigoplus_{k \in \mathbb{N}_+} A^{\otimes(k+1)}$  is a finite sum of elements of the form  $x_0 \otimes \dots \otimes x_n$ ,  $n \in \mathbb{N}^+$ , and a finite sum of nilpotent elements is still nilpotent, we have  $\bigoplus_{k \in \mathbb{N}_+} A^{\otimes(k+1)} \subseteq N(\mathbb{I}\mathbb{I}\mathbb{I}_C(A))$ . Therefore,

$$N(\mathbb{I}\mathbb{I}\mathbb{I}_C(A)) \supseteq N(A) \oplus \left( \bigoplus_{n \in \mathbb{N}_+} A^{\otimes(n+1)} \right).$$

Now let  $x \in \mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  be nilpotent.  $x$  can be uniquely expressed as  $\sum_{k \in \mathbb{N}} x_k$  with  $x_k \in A^{\otimes(k+1)}$ . Since  $x$  is nilpotent,  $x^n = 0$  for some  $n \in \mathbb{N}^+$ . By the definition of the mixed shuffle product  $\diamond$  in  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ , defined in Eq(2), we can uniquely express  $x^n$  as  $\sum_{k \in \mathbb{N}} y_k$  with  $y_k \in A^{\otimes(k+1)}$  and  $y_0 = x_0^n$ . By the uniqueness of  $y_0$ , we have  $y_0 = 0$ . So  $x_0$  is nilpotent. This shows that  $x$  is in  $N(A) \oplus \left( \bigoplus_{n \in \mathbb{N}_+} A^{\otimes(n+1)} \right)$ .

By the same argument as in the last paragraph, we also obtain

$$N(\widehat{\mathbb{I}\mathbb{I}\mathbb{I}}_C(A)) \subseteq N(A) \times \left( \prod_{n \in \mathbb{N}^+} A^{\otimes(n+1)} \right).$$

This completes the proof of 2.

3. We first make a general remark on the mixable shuffle product. For any two tensors  $a_0 \otimes \dots \otimes a_m$  and  $b_0 \otimes \dots \otimes b_m$  of  $A^{\otimes(m+1)}$ , by the definition of the shuffle product in  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$ , we can write

$$(a_0 \otimes \dots \otimes a_m) \diamond (b_0 \otimes \dots \otimes b_m) = \sum_{i \geq m} x_i, \quad x_i \in A^{\otimes(i+1)},$$

and

$$x_m = \lambda^m a_0 b_0 \otimes \dots \otimes a_m b_m.$$

In other words,

$$x_m = \lambda^m (a_0 \otimes \dots \otimes a_m) \cdot (b_0 \otimes \dots \otimes b_m),$$

where we use  $\cdot$  to denote the product in the tensor product algebra  $A^{\otimes(m+1)}$ . By the biadditivity of the multiplication in  $\mathbb{I}\mathbb{I}\mathbb{I}_C(A)$  and the multiplication  $\cdot$

in  $A^{\otimes(m+1)}$ , we see that for any non-zero elements  $a, b \in A^{\otimes(m+1)}$ , the term of  $a \diamond b \in \text{III}_C(A)$  with degree  $m$  equals  $\lambda^m(a \cdot b)$ .

Now let  $x$  be a non-zero element of  $\text{III}_C(A)$ . Express  $x$  as

$$x = \sum_{i=m}^n x_i, \quad x_i \in A^{\otimes(i+1)}, 0 \leq m \leq i \leq n$$

with  $x_m \neq 0$  and  $x_n \neq 0$ . It follows from the above remark and the induction that, for any  $k \geq 1$ , if we express

$$x^k = \sum_{i \geq m} y_i, \quad y_i \in A^{\otimes(i+1)}, i \geq m,$$

then  $y_m = \lambda^{m(k-1)} x_m^{\bullet k}$ . Here  $x_m^{\bullet k}$  stands for the  $k$ -th power of  $x_m$  in the tensor product algebra  $A^{\otimes(m+1)}$ . If  $\lambda$  does not annihilate any non-zero elements in  $A^{\otimes(m+1)}$ , then  $\lambda^m x_m \neq 0$ . If  $A^{\otimes(m+1)}$  is reduced, then we further have  $(\lambda^m x_m)^{\bullet k} \neq 0$ . Since  $(\lambda^m x_m)^{\bullet k} = \lambda^m y_m$ , we have  $y_m \neq 0$ . Then  $x^k$  is not zero, proving that  $\text{III}_C(A)$  has no non-zero nilpotent elements, hence is reduced.

The same argument can be applied to  $\widehat{\text{III}}_C(A)$ , proving that  $\widehat{\text{III}}_C(A)$  is reduced. ■

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